DIMENSIONAL RELATIONS IN MAGNETOHYDRODYNAMICS

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Symbols. Cosmic fluids are as a rule highly turbulent. This entails the necessity of dimensional order-of-magnitude considerations preceding and often even replacing, a more rigorous dynamical theory. We shall here use the symbol $\{a\}$ to designate the order of magnitude of a physical quantity a. In particular, $\{\lambda\}$ will designate a representative length and $\{\omega\}$ will designate a representative reciprocal time. We shall use the rationalized mks. system of units, whence $\mu\varepsilon = c^{-2}$ with the usual meaning of these, and other, electromagnetic symbols. For simplicity it will be assumed that the electrical conductivity, σ, is constant throughout the fluid; ε and μ will be assumed constant throughout space. In vector equations all vectors will be designated by Roman capital letters and scalars by greek or lower case roman letters.

Fixed frame of reference. In large-scale electrodynamics the electromagnetic effects arising from the difference in mass of the positive and negative carriers might not always be negligible. We shall here ignore this type of effect and assume that all the effects considered can be described classically, namely, by a combination of Maxwell's equations with the hydrodynamic equations. Hence V will designate throughout the material velocity of the fluid in a given frame of reference. As a general rule this velocity is small compared to the velocity of light, that is

$$\{\beta\} \ll 1 \tag{1}$$

If a conductor moves across a magnetic field, there appears an induced electric field of magnitude $V \times B$. From the

electromagnetic field equations we then have

$$\mu^{-1} \nabla \times B = J = \epsilon \dot{E} + \sigma E + \sigma V \times B + \eta V \qquad (2)$$

The terms on the right represent in turn, the displacement current, the conduction current, the induction current, and the convection current. The convection current appears owing to the fact that (as we shall see later) E cannot in general be assumed divergence-free and hence there is a space-charge density, η, in the fluid.

The ratio of displacement current to conduction current is of the order

$$\{\omega \epsilon / \sigma\} = \{\gamma\} \qquad (3)$$

This non-dimensional quantity is well known from the electromagnetic theory of metals. To estimate it here we remember that ω now represents frequencies of the macroscopic motion of the fluid. Let us take $\omega = 10^{-4}$ corresponding roughly to periods of a day. Cosmis fluids are as a rule excellent conductors, of metallic order. For the earth's core the conductivity has been estimated to be a factor of 10-100 below that of iron[1]. The material of stars is highly ionized and the conductivities are again of metallic order[2]. Clouds of ionized gases near stars or in interstellar space show as a rule appreciable ionization; they are then again comparable to metallic conductors. (This results from the fact that while the number density of ions becomes small, the mean free path becomes large in the same proportion). Taking as an example $\sigma = 10^7$, the conductivity of ordinary iron, we have $\gamma = 10^{-22}$ which is small indeed.

From the field equations we have $\{\eta\} = \{\epsilon \lambda^{-1} E\}$, hence the ratio or the convection current to conduction current is

$$\{\eta V/\sigma E\} = \{\varepsilon\lambda^{-1}V/\sigma\} = \{\mathscr{A}\} \tag{4}$$

and the convection current is also negligible; (2) reduces to

$$\nabla \times B = \mu \quad = \mu\sigma E + \mu\sigma V \times B \tag{5}$$

We next compare the induction current to the net current. The ratio is

$$\{\mu\sigma VB\} / \{\lambda^{-1}B\} = \{\mu\sigma\lambda V\} = \{R_m\} \tag{6}$$

where R_m is a non-dimensional quantity which will be designated as the magnetic Reynolds number. To elucidate the physical meaning, or one physical meaning, of this quantity we notice that if a current flows in a <u>rigid</u> conductor of linear dimensions λ and conductivity σ the period of free decay, in the absence of an impressed e.m.f., is of the order

$$\{\omega_{dec}^{-1}\} = \{\mu\sigma\lambda^2\} \tag{7}$$

Again, the periods of the material motion of the fluid are of the order $\{\omega_{vel}\} = \{V\lambda^{-1}\}$. Hence

$$\{R_m\} = \{\omega_{vel} / \omega_{dec}\} \tag{8}$$

This relation indicates that R_m is a measure of the coupling between the mechanical motion and the electromagnetic field: As we shall see, the presence of the term $V \times B$ in (5) implies that, in the absence of free decay, the fluid carries the magnetic field along in its motion. The decay phenomena may be visualized as a "diffusion" of the field across the conductor. Strong coupling, including the important case of amplification of the field, requires that the transport of the field by the motion exceeds the rate of diffusion. The distinctive property of cosmic

magnetohydrodynamics becomes evident; in the laboratory R_m is always small; in problems of cosmic hydrodynamics R_m is as a rule very large. The relation of laboratory phenomena to cosmic phenomena of magnetohydrodynamics is, in a rough analog, that of Poisseuille flow to the large scale eddy motions in a star or cosmic cloud. Here, we shall essentially confine ourselves to the cosmic case, that is to large numerical values of the magnetic Reynolds number, and shall not enter into a discussion of the laboratory experiments[3] on the interaction of sound waves with a magnetic field. In order to estimate R_m for the earth's core we take, say $\sigma = 10^6$, $V = 10^{-3}$ m/sec from observations of the secular magnetic variations[1], and $\lambda = 10^6$ meters, giving $R_m = 10^3$. For extra-terrestrial phenomena λ and V are larger by many powers of ten and R_m is correspondingly larger.

Returning now to (5) we see that the net current is negligibly small in large-scale fluids; we have a balance

$$E \approx - V \times B \tag{9}$$

in an excellent approximation. It must not, however, be concluded from (5) that $\nabla \times B$ is negligible in other connections; we shall see for instance that the ponderomotive forces exerted by the field, which depend on $\nabla \times B$, are by no means small.

A further dimensionless quantity of interest is the ratio of the electric to the magnetic energy density. This is

$$\left\{ \epsilon E^2 / \mu^{-1} B^2 \right\} = \left\{ E^2 / c^2 B^2 \right\} = \left\{ \beta^2 \right\} \tag{10}$$

as may be seen from the field equation

$$\nabla \times E = -\partial B / \partial t \tag{11}$$

or else directly from (9).

Taking the curl of (5) we obtain by virtue of (11)

$$\partial B/\partial t = \nabla \times (V \times B) + \nu_m \nabla^2 B \tag{12}$$

where the quantity

$$\nu_m = (\mu\sigma)^{-1} \tag{13}$$

will be designated as the magnetic viscosity. We have $R_m = \lambda V/\nu_m$ which shows that the magnetic Reynolds number is obtained from the ordinary hydrodynamic Reynolds number by replacing ν, the kinematic viscosity, by the magnetic viscosity, ν_m. It will appear more clearly later that ν and ν_m correspond to analogous physical effects.

The integration of (12) is as a rule prohibitively difficult. The physical meaning is brought out more clearly by a corresponding integral theorem[4],

$$\frac{d}{dt} \int B_n ds = -\sigma^{-1} \int J \cdot dL \tag{14}$$

where the surface integral on the left is thought of as moving bodily with the fluid. The contour integral on the right becomes small as R_m becomes large; in the limit of infinite conductivity we obtain the well known result that the magnetic lines of force are "frozen" in the fluid and are carried along with its motion.

From (9) we may infer that the electrical space charge does not in general vanish[5], since

$$\eta/\varepsilon = \nabla \cdot E = V \cdot \nabla \times B - B \cdot \nabla \times V \tag{15}$$

The ratio of the electric to the magnetic components of the electromagnetic stress tensor is, however, given by (10) and the electrostatic forces are negligible. Furthermore it may be shown that the irrotational part of the current (5) is small

compared to the divergence-free part so that the magnetic effects corresponding to a non-vanishing η are also likely to be in general negligible. We have from the equation of continuity for the current, on using (15)

$$\nabla \cdot J = -\dot{\eta} = \left\{ \omega \epsilon E / \lambda \right\} = \left\{ \omega \epsilon VB/\lambda \right\} = \left\{ \omega^2 \epsilon B \right\}$$

and on the other hand

$$\nabla \times J = \mu^{-1} \nabla \times \nabla \times B = \left\{ B/\mu \lambda^2 \right\}$$

Hence it follows that

$$\left\{ \nabla \cdot J \, / \, \nabla \times J \right\} = \left\{ \omega^2 \lambda^2 \mu \epsilon \right\} = \left\{ V^2 \mu \epsilon \right\} = \left\{ \beta^2 \right\} \tag{16}$$

which is small. Hence we conclude that for large magnetic Reynolds numbers we may, without loss of essential physical features, assume η as negligible. This can most conveniently be expressed by introducing a vector potential while dropping the corresponding scalar potential, thus

$$B = \nabla \times A, \quad E = -\partial A/\partial t, \quad \nabla \cdot A = 0 \tag{17}$$

which transforms (5) into

$$\partial A/\partial t = V \times (\nabla \times A) + \nu_m \nabla^2 A \tag{18}$$

an equation that is somewhat simpler than (12).

Lorentz Transformation. In view of (1) we may neglect all terms of the order of β^2 and higher terms. Texts on relativity[6] indicate that σ must be considered as an invariant; this follows from its connection with thermodynamical quantities which are invariant. To within terms linear in β the Lorentz transformation from an unprimed system to a primed system moving with velocity U reduces to

$$R' = R - Ut, \quad t' = t$$

$$\nabla' = \nabla, \quad \partial/\partial t' = \partial/\partial t + U \cdot \nabla \tag{19}$$

Furthermore, in this approximation

$$J' = J + \eta U, \quad \eta' = \eta$$

but the convection current may be neglected if $U = V$. The field vectors transform in the same approximation as

$$E' = E + U \times B, \quad B' = B - U \times E/c^2$$

but the last term in the equation for B' is by (9) of the order β^2 and hence negligible. The transformation equations for the electromagnetic quantities thus reduce to

$$E' = E + U \times B$$

$$B' = B, \quad J' = J, \quad \eta' = \eta \tag{20}$$

We have been brief in this deduction, but it should be emphasized that, as closer consideration shows, all terms linear in β have indeed been included.

We see from (5) that if we transform to a frame of reference in which a given fluid particle is at rest, then the "local" electric field becomes small compared to the average value of E over the fluid, for which (9) gives $\{E\} = \{VB\}$. This latter relation, by the way, permits a convenient observational evaluation of the field, since V and B are quite directly measurable, B being also lorentz-invariant to within terms of the order of β. The actual current referred to a "local" system of reference is therefore small, as R_m^{-1}, compared to the current in, say an engineering dynamo. In such a machine the current is $\{\sigma E\} = \{\sigma VB\}$. Considerable semantic difficulties

are bound to arise when one speaks uncritically of the "electric currents" producing the magnetic fields of the earth, of sunspots, etc. It has been suggested in the literature[7] that since at "neutral" points (points where B vanishes in the "local" frame of reference) charged particles do not travel in spirals as they do elsewhere, phenomena of the type observed in gas discharges might occur which would lead to the acceleration of particles. The smallness of the "local" electric field makes this conclusion unlikely.

Mechanical motion; Symmetrization. The density of the ponderomotive force which the field exerts upon the fluid is

$$F = J \times B = -\mu^{-1} B \times (\nabla \times B)$$
$$= \mu^{-1}(B \cdot \nabla)B - (2\mu)^{-1}\nabla(B^2)$$

(21)

Here, the electrostatic forces produced by the space charges have been neglected since they are small by (10). The work done on the fluid by these forces per unit time and unit volume is $V \cdot F$; it may be shown[4] by obtaining the energy integral from (5) and (11) that this is indeed the negative of the work done by the fluid on the field.

In writing down the equations of motion we shall for simplicity assume that the fluid is incompressible. The equations of motion are

$$\partial V/\partial t + (V \cdot \nabla)V = -\nabla \psi - (\mu\rho)^{-1}(B \cdot \nabla)B + \nu\nabla^2 B$$

(22)

where

$$\rho\psi = p + u + (2\mu)^{-1}B^2$$

and u is the gravitational potential. The equations (12) and

(22) together are the fundamental equations of field-motion of magnetohydrodynamics. They can be put into a remarkably symmetrical form[8)5)]. Letting

$$P = V + (\rho\mu)^{-1}B, \quad Q = V - (\rho\mu)^{-1}B$$

$$\nu_1 = \nu + \nu_m \quad , \quad \nu_2 = \nu - \nu_m$$

and remembering that for an incompressible fluid

$$\nabla_x (V \times B) = (B\cdot\nabla)V - (V\cdot\nabla)B$$

we can rewrite (12) and (19) as

$$\partial P/\partial t + (Q\cdot\nabla)P = -\nabla\psi + \nabla^2(\nu_1 P + \nu_2 Q)$$
$$\partial Q/\partial t + (P\cdot\nabla)Q = -\nabla\psi + \nabla^2(\nu_2 P + \nu_1 Q) \tag{23}$$

where now

$$\psi = p/\rho + u/\rho + (P-Q)^2/8 \tag{24}$$

It should be noted that the symmetrized equations hold only for an incompressible fluid; no corresponding symmetrical formalism has as yet been found for the compressible case. The equations suggest strongly that if magnetohydrodynamics is considered from a __statistical__ view point, as seems appropriate for turbulent fluids, the vectors V and $(\mu\rho)^{-1}B$ should play comparable roles. Energy transfer is possible both from and to the fluid. Many authors have therefore inferred that __equipartition__ of the energy as between the kinetic energy of the fluid and the magnetic field energy might be assumed to hold in a first approximation:

$$\left\{\rho V^2\right\} = \left\{B^2/\mu\right\} \tag{25}$$

Batchelor[9)] has, however, pointed out that if the statistical theory of turbulence is applied to magnetohydrodynamics, the

magnetic field energy should be less than the equipartition value (25), at least for the largest eddies. The author hopes to show elsewhere that under certain conditions the field energy can also exceed the value given by (25). For rough estimates, however, (25) should be useful. An equivalent statement is clearly that in (22)

$$(\mathbf{v} \cdot \nabla) \mathbf{v} \quad = \quad (\mu \rho)^{-1} \, (\mathbf{B} \cdot \nabla) \mathbf{B} \tag{26}$$

the ponderomotive forces are in the mean comparable to the inertial forces. Schlüter and Biermann[10] have pointed out that if the "frictional" term in (12) is neglected this equation is of the type

$$\{\partial B / \partial t\} = \{\lambda^{-1} v\} \{B\}$$

and that the solutions of this equation are of the general form

$$\{B\} = \{B_o\} \{\exp(\lambda^{-1} v) t\}$$

Therefore, if a small magnetic stray field exists in a conducting fluid, it will in the average be amplified at an exponential rate until some statistical equilibrium value near (25) is reached.

Dissipation . The quantity

$$\nu / \nu_m = R_m / R = \mu \sigma \nu \tag{27}$$

measures the ratio of the generation of heat by viscous friction in the fluid to the generation of Joule's heat by the electromagnetic field. Unless this ratio happens to be close to unity, one form of dissipation will as a rule predominate. This form of dissipation will also determine the cutoff of the turbulence spectrum at the side of the smallest eddies.

We can obtain an estimate of (27) for the case of an ionized gas. From kinetic theory we have $\nu = v_0 l_p /3$ and $\sigma = \alpha e^2 N l_p /2mv$ where l_p is the mean free path, α the degree of ionization, N the number density, m and v are mass and mean velocity of the electrons, v_0 mean velocity of the molecules. Since $l_p = (\pi N \partial^2)^{-1}$ where ∂ is the collision diameter, and $v_0/v = (m/m_0)^{1/2}$ we can write this

$$\mu\sigma\nu = \frac{\alpha\mu e^2}{6\pi^2}\left(\frac{m_0}{m}\right)^{1/2}\rho^{-1}\partial^{-4} \tag{28}$$

Consider hydrogen and let (mks units) $\partial = 10^{-10}m$, then

$$\mu\sigma\nu = 2.10^{-4}\alpha/\rho \tag{29}$$

This result shows that in interstellar gas clouds where ρ is very small (10^{-21}mks) the dissipation is entirely caused by mechanical friction, whereas in the interior of stars where ρ is in excess of unity the dissipation is entirely electromagnetic; the transition domain, $\{\mu\sigma\nu\} = 1$, occurs near the density values obtaining in the photospheres of stars.

One should emphasize that even when the electromagnetic dissipation is numerically large, the quantity $\nu_m = (\mu\sigma)^{-1}$ is not in itself a measure of the rate at which the field is dissipated. Since cosmic fluids are highly turbulent, the actual transport or dissipation of any quantity is determined, not by the molecular coefficients of diffusion but by the corresponding eddy diffusivities which are as a rule very much larger than the former. This applies to scalar properties such as heat as well as to vectorial properties such as momentum, and clearly must apply to the magnetic field in the fluid[9]: The calculation of the free decay for a body as large as the sun[2]

yields decay times longer than the age of the universe. This amounts in essence to a computation of the magnetic Reynolds number; there can be little doubt that the general result of turbulence observations applies to the magnetic diffusivity; the larger the Reynolds number, the more the eddy diffusivity exceeds the molecular diffusivity in order of magnitude. The disappearance of sunspot magnetic fields in the course of a few days or weeks is certainly a matter of eddy diffusion.

Electromagnetic potentials. We shall now make some applications of our results to the acceleration of individual particles in conducting fluids. It must be assumed that the particles have a certain initial velocity such that the increase of their kinetic energy by electromagnetic accelerations can exceed the average losses by ionization, collision processes, and radiation; in other words we must assume that an "injection" process[12] exists. Let us inquire into the electromagnetic potentials that accompany magnetohydrodynamic phenomena. On account of the very high energies encountered in cosmic-ray particles it is often presumed that there exist special mechanisms which increase the field strengths, e.g., self-amplificatory plasma oscillations. From the viewpoint of magnetohydrodynamics we might classify as instabilities any processes leading to electrical potential very much in excess of those found in ordinary conducting fluids. Such processes as well as the cyclotron or betatron mechanisms where a particle circulates in the same field many times, are open to the criticism that on purely statistical grounds they are not likely to be sufficiently

widespread or effective: Given the relatively large energy density of the cosmic radiation one is prejudiced in favor of processes that can be counted upon to occur regularly in large volumes of cosmic space.

Here, we shall abstract from all processes except those directly related to the <u>average</u> conditions of cosmic magneto-hydrodynamics. The electrical potential, ϕ, between two points of space is by (9) of the order

$$\{\phi\} = \{\lambda E\} = \{\lambda VB\} \tag{30}$$

If we assume that equipartition prevails this becomes, by (25)

$$\{\phi\} = \{(\mu\rho)^{1/2}\lambda V^2\} \tag{31}$$

These relations must be interpreted with some care. The potentials are of course to be understood as line integrals, $\int Ed\lambda$, along some possible trajectory. The particles spiral along the magnetic lines of force, but these lines of force are not in general closed[13] and the particles will not in general follow the lines accurately owing to collisions and accelerations. Everything depends on the measure (in a set-theoretical sense) of trajectories that actually yield potential differences of the order indicated. If this measure is not too small some particles will be accelerated provided they have the required injection velocities. The above formulas do not discriminate between the non-divergent and the irrotational part of E, the two being of comparable order.

If the region where the acceleration occurs is highly inhomogeneous we may apply (30): In the envelope of a star the density changes very rapidly with height and the magnetic field

will as a rule not be of local origin but will emanate from the lower layers of the star. If we can estimate $\{B\}$ from other data, (30) gives an estimate of the order of magnitude of the accelerating potentials. It is well known that at the occasion of solar flares the sun has ejected numerous particles with energies of the order of 10^8eV. The acceleration of particles in stellar envelopes has been extensively discussed and we may be satisfied to refer to the literature[14]. With $\lambda = 10^9$m, $V = 10^4$m/sec and $B = 10^{-2}$ (= 100 gauss) we obtain $\phi = 10^{11}$ volts. Apart from possible phenomena of instability, it is not likely that magnetohydro-dynamic processes in the neighborhood of stars will lead to potentials exceeding this value by several powers of ten.

If we next consider the gaseous interstellar medium we may assume that the equipartition formula (25) applies, as has been done by a number of authors[12][14]; we may then use (31). The variations in V admissible here are rather limited; $V = 10$ km/sec should be reasonably close to an upper limit. Extremely high voltages could be produced by increasing λ. If we assume one proton per cm^3 in the average over the galaxy, that is (in mks units) $\rho = 10^{-21}$, take $V = 3 \cdot 10^3$ and $\lambda = 10^{20}$ (comparable to the dimensions of the more condensed parts of the galaxy) we obtain $\phi = 10^{14}$ volts. We can increase this value by increasing λ still further. Chandrasekhar and Fermi[15] suspect the presence of magnetic fields in the spiral arms of the galaxy. E. N. Parker[16] has studied the formation of galaxies from an intergalactic gaseous medium and concludes that turbulent velocities of the order of 40 km/sec ought to be present in this medium. If there is also some magnetic field in these dimensions,

it seems possible to account quantitatively for the presence of extremely energetic particles by acceleration over sufficiently large linear dimensions.

If the galactic magnetic field is of the order given by the equipartition formula (25), the particles circulate in the galaxy for a very long time and they will acquire their high energies by <u>multiple</u> interaction with the irregular magnetic field of the gas. The mechanism proposed by Fermi[12] represents a specific model where the magnetohydrodynamic field takes the form of statistical motions of individualized gas clouds. If from the above figures we compute the energy density of the galactic medium it is found slightly in excess of 10^{-13} erg/cm^3. Given the roughness of the data this is very close to the energy density of the cosmic radiation estimated[14] as 10^{-12} erg/cm^3. It is likely, therefore, that the cosmic radiation is nearly in dynamical equilibrium with the galactic magnetic fields and hence also with the motions of the gas. Fan[17] has indicated that one can account for many features of the observed cosmic-ray spectrum on the basis of Fermi's accelerating mechanism. The general principles of magnetohydrodynamics as outlined above give strong support to the idea that the spectral distribution of the cosmic rays must, at least asymptotically, correspond to an equilibrium with the interstellar gas.

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